JOURNAL OF ALGEBRA 19, 329–343 (1971)

Orders in QF and QF 2 Rings

C. R. HAJARNAVIS*

School of Mathematics, The University, Leeds, England

Communicated by A. W. Goldie

Received April 1, 1970

INTRODUCTION

In this paper we shall work with the 'classical' definition of QF rings given in terms of annihilators. In Section 2 we prove the existence of an identity element for a class of rings with minimum condition. A corollary to this is the well-known result that a QF ring has an identity. Necessary and sufficient conditions for a Noetherian ring to have a QF 2 quotient ring are obtained in Section 3 and these are used in Section 4 to determine the (two-sided) orders in QF rings.

Results of these sections are applied to study the structure of Noetherian rings which are expressible as direct sums of uniform right ideals and also of uniform left ideals (Condition (T)); and have zero singular ideal. In Section 6 it is shown that a ring satisfying the above conditions is expressible as a direct sum of irreducible rings each satisfying the same conditions. Uniqueness of this decomposition is shown in Section 7.

Artinian rings satisfying condition (T) are just the QF 2 rings. These have been studied in [1, 4, 7]. The concept of Basic right ideal is due to Goldie [7].

Some Definitions and Notation

The term ring will mean an associative ring not necessarily commutative. We do not assume the existence of an identity element in the first four sections.

Let R be a ring. We shall write:

$N(R)$ = the maximal nilpotent ideal of R. (This exists in all the cases we consider here).
$Z(R)$ $[Z'(R)]$ = the right [left] singular ideal of R.
$E(R)$ $[E'(R)]$ = the right [left] socle of R.

* Most of the results of this paper are included in the author's Ph.D. thesis (Leeds).

If S is a subset of R, we shall write $r_R(S)$ and $l_R(S)$ to denote the right annihilator and the left annihilator of S in R. If no confusion is likely to arise, we may drop the argument or suffix 'R'.

We shall write $\dim_r R$ and $\dim_l R$ to denote the dimension of R as a right R-module and as a left R-module, respectively.

If I is an ideal of R then $\mathscr{C}'(I)$, $'\mathscr{C}(I)$ and $\mathscr{C}(I)$ will denote $c \in R$ such that the coset $c + I$ is, respectively, right regular, left regular, and regular in the ring R/I.

An ideal [right ideal, left ideal] is *indecomposable* if it cannot be expressed as a nontrivial direct sum of ideals [right ideals, left ideals]. Further, if an indecomposable right [left] ideal is idempotently generated, we say that it is a *primitive* right [left] ideal. The ring R is indecomposable if it is indecomposable as an ideal. An *irreducible* ideal [right ideal] is one which cannot be expressed as a nontrivial intersection of two ideals [right ideals]. R is an irreducible ring if 0 is an irreducible ideal.

Unless otherwise stated, conditions will be assumed to hold on both sides, e.g., Noetherian will mean left and right Noetherian.

1. Preliminaries

The reader is reminded of some useful results.

LEMMA 1.1. *Let R be a (right) finite-dimensional ring. Let M be a maximal complement in R. Then M is an irreducible right ideal and every right ideal properly contained in M is reducible.*

Proof. See [5], p. 43.

LEMMA 1.2. *Let R be a right Goldie ring with $Z(R) = 0$. Then*

(i) *If U, X are right ideals of R such that $U \cap X \neq 0$ and U is a uniform right ideal then $U \subseteq \operatorname{cl} X = \{x \in R \mid xE \subseteq X$ for some essential right ideal $E\}$.*

(ii) *For any right ideal I, $\dim I = \dim \operatorname{cl} I$.*

(iii) *If U is a uniform right ideal and θ is an R-hom of $U \to R$, then θ is either zero or an isomorphism into R.*

Proof. (i) See [5, p. 45]. (ii) See [5, Theorem 3.8]. (iii) See [7, Lemma 5.4].

DEFINITION. (i) Two right ideals U and V of a ring R are said to be *subisomorphic* if there exist R-isomorphisms θ, ϕ such that $U\theta \subseteq V$ and $V\phi \subseteq U$.

(ii) A right ideal U of R is said to be *basic* if it is subisomorphic to all the nonzero right ideals contained in U.

Clearly, in a (right) finite-dimensional ring a basic right ideal is uniform.

LEMMA 1.3. *Let R be a right Noetherian ring with $Z(R) = 0$.* Then

(i) *Every nonzero right ideal of R contains a basic right ideal.*

(ii) *If U is a uniform right ideal such that $U \cap N = 0$ then U is a basic right ideal.*

Proof. (i) See [7, Theorem 3.6]. (ii) See [7, p. 276].

LEMMA 1.4. *Let eR be an idempotently generated uniform right ideal of a ring R. Then $r(eR)$ is an irreducible ideal.*

Proof. Let A, B be ideals of R such that $r(eR) = A \cap B$. Then $eA \cap eB = 0$, and so $A = r(eR)$ or $B = r(eR)$.

LEMMA 1.5. *Let R be a right Goldie ring with $Z(R) = 0$. Let I be a right ideal of R and suppose $d \in R$ is regular. Then $r(I) = r(Id)$.*

Proof. See [3, Proprieté 2.6].

THEOREM 1.6. *Let R be a right Noetherian ring. Then*

(i) *(Goldie) $\mathscr{C}'(0) \subseteq \mathscr{C}(N)$ in R.*

(ii) *(Small) R has a right Artinian right quotient ring if and only if $\mathscr{C}(N) \subseteq \mathscr{C}(0)$.*

Proof. (i) See [6, Theorem 1.5]. (ii) See [6, Theorem 1.7].

LEMMA 1.7. *Let R be a right Noetherian ring with right quotient ring Q. Let $N = N(R)$, $N' = N(Q)$. Then $(N')^k = N^k Q$ for $k = 1, 2, \ldots$.*

Proof. See [6, Corollary 1.6].

LEMMA 1.8. *Let eQ, fQ be two primitive right ideals of a right Artinian ring Q. Then $eQ \cong fQ$ if and only if $r(eQ) = r(fQ)$.*

Proof. See [4, p. 10].

LEMMA 1.9. *Let U be a uniform right ideal of a right Artinian ring Q. Let M be the unique minimal right ideal contained in U. Suppose A is a right ideal of Q such that $AM \neq 0$. Then $U \cong A_0$ where A_0 is a right ideal contained in A.*

Proof. $aM \neq 0$ for some $a \in A$. Let θ be the Q-hom $u \to au$ for all $u \in U$. If $\ker \theta \neq 0$ then $M \subseteq \ker \theta$ and $aM = 0$ which is a contradiction. Hence $\ker \theta = 0$ and $U \cong aU = A_0 \subseteq A$.

LEMMA 1.10. *Let R be a right Noetherian ring with 1. Then R can be expressed as a direct sum of indecomposable ideals and this expression is unique.*

2. THE IDENTITY ELEMENT

LEMMA 2.1. *Let R be a ring with the minimum condition on right ideals. Then $E' \subseteq r(N)$.*

Proof. If L is a minimal left ideal of R, then we have either $NL = 0$ or $NL = L$. Since N is nilpotent the latter possibility does not occur. Hence $NE' = 0$.

THEOREM 2.2. *Let R be a ring which has the minimum condition on right ideals. Suppose R also satisfies*

(i) $l(R) = r(R) = 0$,

(ii) $E \subseteq E'$.

Then R has an identity element.

Proof. Since $l(R) = 0$, we have $R \neq N$. Therefore R/N is a (nontrivial) semisimple Artinian ring. Let $c + N$ be the identity element of R/N. We have $R = Rc + N$. Hence $r(Rc) \cap r(N) = r(Rc + N) = r(R) = 0$. Therefore $r(Rc) \cap E' = 0$. Hence $r(Rc) \cap E = 0$. This implies $r(c) = 0$.

Now, since R has the minimum condition on right ideals, there exists an integer n such that $c^n R = c^{n+1} R$. Hence there exists $e \in R$ such that $c^n c = c^{n+1} e$. Since $r(c) = 0$, we have $c = ce$. For any $x \in R$, $c(x - ex) = 0$ so $x = ex$. Now take any arbitrary $y \in R$. We have $(x - xe)y = xy - x(ey) = 0$. Hence $(x - xe)R = 0$. Since $l(R) = 0$, we must have $x = xe$. Therefore, $x = ex = xe$ for all $x \in R$. Thus e is the identity of R.

DEFINITION. A ring R with the minimum condition on the left ideals as well as the right ideals is said to be a *QF ring* if every right ideal of R is a right annihilator and every left ideal is a left annihilator.

Clearly, then, a QF ring also has the maximum condition.

LEMMA 2.3. *Let R be a QF ring. Then*

(i) $l(R) = r(R) = 0$.

(ii) $E = l(N) = E' = r(N)$.

Proof. (i) By definition of a QF ring we have, $0 = lr(0) = l(R)$, and $0 = rl(0) = r(R)$.

(ii) $E \subseteq l(N)$ and $E' \subseteq r(N)$ as in 2.1. Since R has the maximum condition on right ideals, $Z = l(E) \subsetneq N$ by Levitzki's theorem. Therefore, $r(N) \subseteq rl(E) = E$. Thus $E' \subsetneq r(N) \subseteq E$. Similarly, $E \subseteq l(N) \subseteq E'$. Hence $E = l(N) = E' = r(N)$.

COROLLARY 2.4. *A QF ring has an identity element.*

Proof. Follows from 2.2 and 2.3.

3. ORDERS IN QF 2 RINGS

DEFINITION. An element u of a ring R is said to be *right uniform* if uR is a uniform right ideal.

LEMMA 3.1. *Let R be a right Noetherian ring. Let $u \in R$ be a right uniform element such that $u \notin N$. Then $r(u)$ is a maximal complement in R.*

Proof. Let $\dim_r R = n$. We shall first show that for any $a \in R$, $\dim aR + \dim r(a) \geqslant n$. Let $U_1 \oplus \cdots \oplus U_k$ be a direct sum of uniform right ideals which is essential in $r(a)$. Let K be a right complement of $r(a)$ in R. Choose $U_{k+1}, \ldots, U_n$ uniform such that $U_1 \oplus \cdots \oplus U_n$ is a direct sum. It is easily checked that $aU_{k+1} + \cdots + aU_n$ is also a direct sum and $aU_i \neq 0$ for $k+1 \leqslant i \leqslant n$. Therefore, $\dim aR \geqslant n - k$. Thus $\dim aR + \dim r(a) \geqslant n$. In particular, $\dim r(u) \geqslant n - 1$ since $\dim uR = 1$. If $\dim r(u) = n$; i.e., if $r(u)$ is an essential right ideal then we have $u \in Z \subseteq N$; a contradiction. Hence $\dim r(u) = n - 1$. Let M be a maximal right complement containing $r(u)$. It is easily checked that $r(u)$ is an irreducible right ideal. It follows by 1.1 that $r(u) = M$.

The next lemma is an unpublished result of Goldie.

LEMMA 3.2. *Let R be a right Goldie ring. Let $c \in R$ with $r(c) = 0$. Then $r(c + n) = 0$ for any $n \in N$.*

LEMMA 3.3. *Let R be a right Noetherian ring which has a direct sum of uniform right ideals containing a right regular element. Let I be a right ideal of R which intersects every non-nilpotent right ideal of R in a non-nilpotent right ideal. Then*

$$I \cap \mathscr{C}'(0) \neq \phi.$$

Proof. Let $U_1 \oplus \cdots \oplus U_n$ be a direct sum of uniform right ideals containing $c \in \mathscr{C}'(0)$. (Clearly, then $n = \dim_r R$). Let $c = u_1 + \cdots + u_n$ where $u_i \in U_i$. By 3.2, $u_i \notin N$ for any i and so, $u_iR \nsubseteq N$ for any i. Write

$V_i = I \cap u_iR$. Then $V_i \not\subseteq N$ by assumption. Also, $V_iu_i \not\subseteq N$ since $V_iu_i \subseteq N \Rightarrow V_iu_iR \subseteq N \Rightarrow V_i^2 \subseteq N \Rightarrow V_i \subseteq N$, a contradiction. Hence there exists $v_i \in V_i$ such that $v_iu_i \notin N$. Now, $r(v_iu_i) \supseteq r(u_i)$. Therefore, $\dim r(v_iu_i)$ equals n or $n-1$. If $\dim r(v_iu_i) = n$ then $r(v_iu_i)$ will be an essential right ideal and we shall have $v_iu_i \in Z \subseteq N$ which is a contradiction. Hence $\dim r(v_iu_i) = n-1$. Therefore, $r(v_iu_i) = r(u_i)$ for every i. Now,

$$\begin{aligned} 0 &= r(u_1 + \cdots + u_n) = r(u_1) \cap \cdots \cap r(u_n) \\ &= r(v_1u_1) \cap \cdots \cap r(v_nu_n) \\ &= r(v_1u_1 + \cdots + v_nu_n). \end{aligned}$$

Thus $v_1u_1 + \cdots + v_nu_n \in I \cap \mathscr{C}'(0)$.

LEMMA 3.4. *Let R be a (right) finite dimensional ring. Then for any $c, d \in R$ we have $r(c) = 0 = r(d)$ if and only if $r(cd) = 0$.*

Proof. Suppose $r(cd) = 0$. Then clearly $r(d) = 0$. Suppose $r(c) \neq 0$. Then $dR \cap r(c) \neq 0$ since dR is an essential right ideal of R. Let $x \in dR \cap r(c)$, $x \neq 0$. Then $x = dt$ for some $t \in R$; and $cx = cdt = 0$. Therefore $t = 0$ and hence $x = 0$ which is a contradiction.

The converse is trivial.

The next lemma was suggested by A. Ludgate.

LEMMA 3.5. *Let R be a right Noetherian ring which has a direct sum of uniform right ideals containing a right regular element. Then*

$$\mathscr{C}(N) \subseteq \mathscr{C}'(0) \text{ in } R.$$

Proof. Let $d \in \mathscr{C}(N)$. Then $(dR + N)/N$ is an essential right ideal of R/N. Hence $dR + N$ satisfies the condition on the right ideal I of 3.3. Therefore there exist $t \in R$, $n \in N$ such that $r(dt + n) = 0$. Hence $r(dt) = 0$ by 3.2. Therefore $r(d) = 0$ by 3.4.

LEMMA 3.6. *Let R be a ring with a right quotient ring Q. Then*

(i) *U is a uniform right ideal of $R \Rightarrow UQ$ is a uniform right ideal of Q.*

(ii) *V is a uniform right ideal of $Q \Rightarrow V \cap R$ is a uniform right ideal of R.*

(iii) *$A_1 \oplus \cdots \oplus A_k$ is a direct sum of right ideals in $R \Rightarrow A_1Q + \cdots + A_kQ$ is a direct sum of right ideals in Q.*

(iv) *$B_1 \oplus \cdots \oplus B_m$ is a direct sum of right ideals in $Q \Rightarrow (B_1 \cap R) + \cdots + (B_m \cap R)$ is a direct sum of right ideals in R.*

(v) $\dim_r R = \dim_r Q$.

DEFINITION. A right Artinian ring Q is said to be an *$rQF\,2$ ring* if every primitive right ideal of Q is uniform. An *$lQF\,2$ ring* is defined similarly. If a ring is both $rQF\,2$ and $lQF\,2$, then we call it a *$QF\,2$ ring*.

PROPOSITION 3.7. *Let Q be a right Artinian ring. The following statements are equivalent*:

(1) *Q is an $rQF\,2$ ring.*

(2) *Q is expressible as a sum of uniform right ideals.*

(3) *Q is expressible as a direct sum of uniform right ideals.*

Proof. See [7, Lemma 7.1] where the equivalence of (1) and (2) is shown. It is easy to show that these are equivalent to (3).

COROLLARY 3.8. *If Q is a $QF\,2$ ring then* $\dim_r Q = \dim_l Q$.

Proof. We can express Q as $Q = e_1Q \oplus \cdots \oplus e_nQ = Qe_1 \oplus \cdots \oplus Qe_n$, where the e_iQ are primitive right ideals and the Qe_j are primitive left ideals. Since the primitive one-sided ideals of Q are uniform, the result follows.

THEOREM 3.9. *Let R be a Noetherian ring. Then R is an order in a $QF\,2$ ring if and only if*

(i) *There exists a direct sum of uniform right ideals in R containing a right regular element.*

(ii) *There exists a direct sum of uniform left ideals in R containing a left regular element.*

Proof. Suppose conditions (i) and (ii) hold in R. Then $\mathscr{C}(N) \subseteq \mathscr{C}'(0) \cap {}'\mathscr{C}(0)$ by 3.5 and symmetry. Hence $\mathscr{C}(N) \subseteq \mathscr{C}(0)$. Therefore by 1.6 (ii), R has an Artinian quotient ring. Let $c \in \mathscr{C}(0)$ be such that $c \in U_1 \oplus \cdots \oplus U_n$, where the U_i are uniform right ideals of R. By 3.6, $U_1Q \oplus \cdots \oplus U_nQ$ is a direct sum of uniform right ideals in Q. Since c is a unit of Q, we must have $Q = U_1Q \oplus \cdots \oplus U_nQ$. Hence by 3.7 and symmetry, Q is a $QF\,2$ ring.

Conversely, suppose R has a quotient ring which is a $QF\,2$ ring. Let $Q = V_1 \oplus \cdots \oplus V_n$ where the V_i are uniform right ideals of Q. Let $1 = t_1c_1^{-1} + \cdots + t_nc_n^{-1}$, where $t_i \in V_i \cap R$ and the c_i are regular in R. Hence there is a regular element $c \in (V_1 \cap R) \oplus \cdots \oplus (V_n \cap R)$ which by 3.6 is a direct sum of uniform right ideals in R. Thus condition (i) holds in R. Condition (ii) follows similarly.

COROLLARY 3.10. *Let R be an order in a $QF\,2$ ring. Then* $\dim_r R = \dim_l R$.

Proof. Let Q be the quotient ring of R. Since Q is Artinian, R is (left as well as right) finite dimensional. Hence

$$\begin{aligned} \dim_r R &= \dim_r Q \quad \text{by } 3.6\,(\text{v}) \\ &= \dim_l Q \quad \text{by } 3.8 \\ &= \dim_l R \quad \text{by } 3.6\,(\text{v}). \end{aligned}$$

4. Orders in QF Rings

The connection between QF rings and $QF\,2$ rings is given by the next theorem due to Kupisch [8].

Theorem 4.1. *The following statements are equivalent.*

(i) *Q is a QF ring.*

(ii) *Q is a $QF\,2$ ring and $E(Q) = E'(Q)$.*

Lemma 4.2. *Let R be a Noetherian ring with quotient ring Q. Let $N = N(R)$. Then $l_R(N) = r_R(N)$ if and only if $l_Q(NQ) = r_Q(NQ)$.*

Proof. This depends on the fact that $N(Q) = NQ = QN$. We omit the details.

Theorem 4.3. *Let R be a Noetherian ring. Then R is an order in a QF ring if and only if R satisfies the following three conditions:*

(i) *There exists a direct sum of uniform right ideals in R containing a right regular element.*

(ii) *There exists a direct sum of uniform left ideals in R containing a left regular element.*

(iii) *$l(N) = r(N)$, where $N = N(R)$.*

Proof. Suppose R satisfies the given conditions. By 3.9, R has a $QF\,2$ quotient ring, Q say. Condition (iii) and 4.2 imply $l_Q(NQ) = r_Q(NQ)$. Since $N(Q) = NQ = QN$, we have $E(Q) = E'(Q)$ (See [2, Lemma 58.3]). Therefore, by 4.1, Q is a QF ring. The converse is similar.

5. Rings Which Are Direct Sums of Uniform Right [Left] Ideals

From this point onwards, unless otherwise stated, all rings will be assumed to be Noetherian. The existence of an identity element will also be assumed.

DEFINITION. We say that a ring R *satisfies condition* (T) if R is expressible as a direct sum of uniform left ideals as well as uniform right ideals.

Thus, in particular, a $\underset{\sim}{Q}F$ ring satisfies condition (T). We shall use here the results of the previous sections to investigate the structure of rings which satisfy condition (T) and have the right singular ideal zero.

LEMMA 5.1. *Let $\underset{\sim}{Q}$ be a $\underset{\sim}{Q}F2$ ring with $Z(\underset{\sim}{Q}) = 0$. Then $Z'(\underset{\sim}{Q}) = 0$.*

Proof. See [7, Corollary to Lemma 7.2].

THEOREM 5.2. *Let R be a ring satisfying condition (T) with $Z(R) = 0$. Then*

(i) *R has a quotient ring $\underset{\sim}{Q}$, where $\underset{\sim}{Q}$ is a $\underset{\sim}{Q}F2$ ring and $Z(\underset{\sim}{Q}) = 0$.*

(ii) $Z'(R) = 0$.

(iii) *$\dim_l R = \dim_r R$ and R is expressible as $R = e_1R \oplus \cdots \oplus e_nR = Re_1 \oplus \cdots \oplus Re_n$ where the e_i are mutually orthogonal idempotents, $1 = e_1 + \cdots + e_n$, each e_iR is a uniform right ideal and each Re_i is a uniform left ideal.*

Proof. (i) The existence of a $\underset{\sim}{Q}F2$ quotient ring is given by 3.9. Since $Z(R) = 0$, it follows that $Z(\underset{\sim}{Q}) = 0$.

(ii) By 5.1, we have $Z'(\underset{\sim}{Q}) = 0$. Hence $Z'(R) = 0$.

(iii) We can write $R = e_1R \oplus \cdots \oplus e_nR$, where the e_iR are uniform right ideals, $1 = e_1 + \cdots + e_n$, and the e_i are mutually orthogonal idempotents. This gives $R = Re_1 \oplus \cdots \oplus Re_n$. Therefore, $\dim_l R \leqslant n = \dim_r R$. But we also have $\dim_r R \leqslant \dim_l R$ by symmetry. Hence $\dim_l R = \dim_r R$ and each Re_i is a uniform left ideal.

6. EXISTENCE OF DECOMPOSITION

It is easily seen that the basic right ideals contained in a uniform right ideal are all mutually subisomorphic. This enables us to define an equivalence relation on the uniform right ideals of a ring R as follows:

If U, V are uniform right ideals in R then $U \sim V$ if and only if the basic right ideals in U and V are mutually subisomorphic. We shall denote the equivalence class of U by $\{U\}$. A similar equivalence was defined by Goldie in [7].

DEFINITION. If U is a uniform right ideal, define S_U to be the sum of all right ideals in $\{U\}$.

If $Z(R) = 0$, then for any $x \in R$ either $xU = 0$ or $xU \simeq U$ by 1.2 (iii). Thus if $Z(R) = 0$, then S_U is an ideal of R.

We can also make similar definitions on the left.

LEMMA 6.1. *Let R be a ring with $Z(R) = 0$. Let B be a basic right ideal in S_U, where U is a uniform right ideal of R. Then $B \in \{U\}$.*

Proof. Choose $U_1 \oplus \cdots \oplus U_k$ a direct sum of uniform right ideals essential in S_U. Then for any $x \in S_U$ there exists an essential right ideal F such that $xF \subseteq U_1 \oplus \cdots \oplus U_k$. We shall first show that $U_i \in \{U\}$ for each $i = 1,\ldots, k$. Note that $S_U \not\subseteq \mathrm{cl}(U_2 \oplus \cdots \oplus U_k)$ since by 1.3 (iii), dim $\mathrm{cl}(U_2 \oplus \cdots \oplus U_k) = k - 1$ but dim $S_U = k$. Hence there exist an element v in some $V \in \{U\}$ and an essential right ideal F_1 such that $vF_1 \subseteq U_1 \oplus \cdots \oplus U_k$ and the projection of vF_1 into U_1 is not zero. Then for any $z \in vF_1$, we have $z = u_1 + \cdots + u_k$ uniquely, where $u_i \in U_i$. So $z \to u_1$ defines an R-hom of vF_1 into U_1. By 1.2 (iii) it is either zero or an isomorphism into U_1. But we have chosen v such that this map is not zero. Hence it is an isomorphism. Thus $vF_1 \sim U_1$ and so $U \sim V \sim vF_1 \sim U_1$. Therefore, $U_1 \in \{U\}$. Similarly, $U_i \in \{U\}$ for $i = 2,\ldots, k$. Now there are basic right ideals $B_i \subseteq U_i$ such that $B_1 \oplus \cdots \oplus B_k$ is essential in S_U. Choose $z \in B$, $z \neq 0$. There exists F' an essential right ideal such that $zF' \subseteq B_1 \oplus \cdots \oplus B_k$. Since B is basic, there exists an isomorphism θ of B into zF'. Let $b\theta = b_1 + \cdots + b_k$ for any $b \in B$. Then $b \to b_i$ is a homomorphism which cannot be zero for all i. Therefore there exists j such that B is isomorphic to $B' \subseteq B_j$. Also since B_j is basic, there is an isomorphism of B_j into B'. This isomorphism combined with the inverse of the isomorphism of B onto B' gives an isomorphism of B_j into B. Thus B and B_j are subisomorphic. This completes the proof.

LEMMA 6.2. *Let R be a ring with $Z(R) = 0$. If $\{U_1\},\ldots, \{U_r\}$ is any finite set of distinct equivalence classes then the sum of ideals $S_{U_1} + \cdots + S_{U_r}$ is direct. Thus, in particular, there are in all only a finite number of equivalence classes.*

Proof. This follows by induction using 6.1.

LEMMA 6.3. *Let R be an indecomposable ring satisfying condition (T) and $Z(R) = 0$. Then R has only one equivalence class.*

Proof. Since R is a direct sum of uniform right ideals we have $R = S_{U_1} \oplus \cdots \oplus S_{U_r}$ if $\{U_1\},\ldots, \{U_r\}$ are the (distinct) equivalence classes of R. Thus $r = 1$ and R has only one equivalence class.

LEMMA 6.4. *Let R be a ring satisfying condition (T) and $Z(R) = 0$. Let $R = e_1R \oplus \cdots \oplus e_nR$, where the e_iR are uniform right ideals (as in 5.2 (iii)). Then Re_j is a basic left ideal for some j; $1 \leqslant j \leqslant n$.*

Proof. Let ρ be the integer such that $N^\rho = 0$ but $N^{\rho-1} \neq 0$. Then $e_jN^{\rho-1} \neq 0$ for some j. Now $Ne_jN^{\rho-1} = 0$; so $r(N) \cap e_jR \neq 0$. Therefore, $e_jR \subseteq r(N)$ by 1.2 (i) as $r(N)$ is a closed right ideal. Hence $Re_j \cap N = Ne_j = 0$. By 5.2 (iii) Re_j is a uniform left ideal. Therefore, by 1.3 (ii) it follows that Re_j is a basic left ideal.

THEOREM 6.5. *Let R be an indecomposable ring satisfying condition(T) with $Z(R) = 0$. Then R is an irreducible ring.*

Proof. By 6.4 there is an idempotent e in R such that eR is a uniform right ideal and Re is a basic left ideal. Suppose ReR is not essential as a left ideal. Then there exists B a basic left ideal such that $B \cap ReR = 0$. Hence $ReRB = 0$ and $ReR \subseteq l(B)$. Since R is indecomposable, it has only one equivalence class of uniform left ideals by the left handed version of 6.3. (Note that $Z'(R) = 0$ by 5.2 (ii)). Hence Re is subisomorphic to B. Therefore, $l(Re) = l(B)$. This implies $ReRRe = 0$, which is a contradiction since $e \neq 0$. Thus ReR is an essential left ideal, and since $Z'(R) = 0$ we have $r(eR) = r(ReR) = 0$. Now suppose $0 = A \cap B$ where A, B are nonzero ideals of R. Then $eR \cap A \neq 0$ since $eR \cap A = 0 \Rightarrow eRA = 0 \Rightarrow A = 0$. Similarly, $eR \cap B \neq 0$. But $(eR \cap A) \cap (eR \cap B) = 0$. This is a contradiction since eR is a uniform right ideal. Thus R is an irreducible ring.

LEMMA 6.6. *Let R be a ring with right quotient ring Q. If R is irreducible, so is Q.*

Thus to sum up we have the following theorem:

THEOREM 6.7. *Let R be a ring satisfying condition (T) and $Z(R) = 0$. Then R can be expressed uniquely up to ordering as a (finite) direct sum $R = R_1 \oplus \cdots \oplus R_m$ of irreducible rings where each R_i satisfies the same conditions as R and has a quotient ring Q_i which is an irreducible $QF\,2$ ring.*

Proof. The uniqueness of the expression is immediate from 1.10.

7. RELATION BETWEEN TWO DECOMPOSITIONS

It remains to investigate the relation between the above decomposition of R and any decomposition of type given in 5.2 (iii).

Notation 7.1. Suppose $R = e_1R \oplus \cdots \oplus e_nR = Re_1 \oplus \cdots \oplus Re_n$ as in 5.2 (iii). Set $T_i = r(e_iR)$ and renumber so that $T_1, \ldots, T_k$ are the minimal T_i. We shall show that these T_i are invariants of R. Note first that

$$0 = r(R) = r\left(\sum_1^n e_iR\right) = \bigcap_1^n r(e_iR) = \bigcap_1^n T_i = \bigcap_1^k T_i .$$

Our next task is to show that this representation of 0 is irredundant.

LEMMA 7.2. *Let R be a ring with $Z(R) = 0$. Suppose R has a right quotient ring $\underset{\sim}{Q}$. Let I be a right ideal of R. Then*

(i) $r_Q(I\underset{\sim}{Q}) = r_R(I)\underset{\sim}{Q}$,

(ii) $r_R(I) = r_Q(I\underset{\sim}{Q}) \cap R$.

Proof. (i) Let $x \in r_R(I)$ and let $yd^{-1} \in I\underset{\sim}{Q}$, where $y \in I$ and d is regular ni R. Consider the element $yd^{-1}x$. We have $d^{-1}x = zf^{-1}$ for some $z, f \in R$; f regular in R. Therefore, $dz = xf$. Now $Idz = Ixf = 0$. Hence $z \in r_R(Id) = r_R(I)$ by 1.5. Therefore, $yd^{-1}x = yzf^{-1} = 0$. Hence $r_R(I)\underset{\sim}{Q} \subseteq r_Q(I\underset{\sim}{Q})$. The reverse inclusion is clear. (ii) follows from (i).

DEFINITION. A right ideal I of a ring R is said to be a *minimal faithful right ideal* if $r(I) = 0$ and any right ideal properly contained in I has nonzero right annihilator.

Clearly, a right Artinian ring has a minimal faithful right ideal.

The next proposition is due to M. Dwan [4].

PROPOSITION 7.3. *Let $\underset{\sim}{Q}$ be a right Artinian ring. Let I be a minimal faithful right ideal of $\underset{\sim}{Q}$. Then $I = e\underset{\sim}{Q}$ for some idempotent e of $\underset{\sim}{Q}$.*

Proof. I is not nilpotent and so has Peirce decomposition $I = e\underset{\sim}{Q} \oplus V$ where e is a nonzero idempotent and V is nilpotent. There exists integer n such that $V^n = 0$ but $V^{n-1} \neq 0$. Then $V^{n-1} \subseteq r(V)$. Now $0 = r(I) = r(e\underset{\sim}{Q}) \cap r(V)$. Hence $r(e\underset{\sim}{Q}) \cap V^{n-1} = 0$, and so $V^{n-1}r(e\underset{\sim}{Q}) = 0$. This implies $V^{n-2}r(e\underset{\sim}{Q}) \subseteq r(V) \cap r(e\underset{\sim}{Q}) = 0$. Continuing this way, we obtain $r(e\underset{\sim}{Q}) \subseteq r(V)$. It follows that $r(e\underset{\sim}{Q}) = 0$ and hence $I = e\underset{\sim}{Q}$.

PROPOSITION 7.4. *Let R be a ring satisfying condition (T) and $Z(R) = 0$. Let $\underset{\sim}{Q}$ be the quotient ring of R. Then $e_1\underset{\sim}{Q} \oplus \cdots \oplus e_k\underset{\sim}{Q}$ is a minimal faithful right ideal of $\underset{\sim}{Q}$.*

Proof. First note that by 7.2 $r_Q(e_1\underset{\sim}{Q}), \ldots, r_Q(e_k\underset{\sim}{Q})$ are the distinct minimal elements of the set $\{r_Q(e_i\underset{\sim}{Q}) \mid i = 1, \ldots, n\}$. By 7.3, $\underset{\sim}{Q}$ has a minimal faithful right ideal eQ for some idempotent e of Q. We can express eQ as a direct sum of nonisomorphic primitive right ideals. Since Q is Artinian, each of these is isomorphic to some e_iQ; $1 \leqslant i \leqslant n$. Hence, without loss of generality, we can take eQ to be $e_{j_1}Q \oplus \cdots \oplus e_{j_s}Q$ for some e_{j_m}'s in the set $e_1, \ldots, e_n$. Consider e_tQ for some t; $1 \leqslant t \leqslant k$. Let M be the unique minimal right ideal contained in e_tQ. Since $eQM \neq 0$, there exists an integer j_p; $1 \leqslant p \leqslant s$ such that $e_{j_p}QM \neq 0$. Hence by 1.9, e_tQ is isomorphic to some right ideal K contained in $e_{j_p}Q$. Thus $r_Q(e_tQ) \supseteq r_Q(e_{j_p}Q)$. Hence $r_Q(e_tQ) = r_Q(e_{j_p}Q)$, because the first k annihilators are chosen to be minimal. Therefore, $e_tQ \simeq e_{j_p}Q$ by 1.8. Since, by 1.8, no $e_\alpha Q$ is isomorphic to $e_\beta Q$ for $1 \leqslant \alpha, \beta \leqslant k$; $e_1Q \oplus \cdots \oplus e_kQ$ must be isomorphic to a right ideal contained in $e_{j_1}Q \oplus \cdots \oplus e_{j_m}Q$. But $r(e_1Q \oplus \cdots \oplus e_kQ) = 0$ since $r(Q) = 0$. Hence $e_1Q \oplus \cdots \oplus e_kQ$ must be a minimal faithful right ideal of Q.

LEMMA 7.5. *Let R be a ring satisfying condition (T) and $Z(R) = 0$. Then with notation as in* 7.1, *the intersection $\bigcap_1^k T_i = 0$ is irredundant.*

Proof. Suppose say T_1 is redundant so that $\bigcap_2^k T_i = 0$. Therefore $\bigcap_2^k T_iQ = (\bigcap_2^k T_i)Q = 0$. Hence $\bigcap_2^k r_Q(e_iQ) = 0$ by 7.2. i.e., $r_Q(e_2Q \oplus \cdots \oplus e_kQ) = 0$. But by 7.4, $e_1Q \oplus \cdots \oplus e_kQ$ is a minimal faithful right ideal of Q. Thus we arrive at a contradiction and so no T_i is redundant in the intersection.

THEOREM 7.6. *Let R be a ring satisfying condition (T) and $Z(R) = 0$. Let $R = R_1 \oplus \cdots \oplus R_m$ be the decomposition of* 6.7. *Let $R = e_1R \oplus \cdots \oplus e_nR$ be any decomposition as in* 7.1. *Then*

(1) *$k = m$ and the T_i $(1 \leqslant i \leqslant k)$ can be renumbered so that $T_j = R_1 \oplus \cdots \oplus R_{j-1} \oplus R_{j+1} \oplus \cdots \oplus R_m$.*

(2) *The e_i can be renumbered as $e_{ij}(1 \leqslant i \leqslant m)$ so that $R_i = \sum_j e_{ij}R$ $(1 \leqslant i \leqslant m)$.*

Proof. (1) Write $\bar{R}_j = R_1 \oplus \cdots \oplus R_{j-1} \oplus R_{j+1} \oplus \cdots \oplus R_m$. We have $0 = e_j \cdot (\bigcap_1^m \bar{R}_i) = \bigcap_{i=1}^m e_j\bar{R}_i$. Since e_jR is a uniform right ideal, we must have at least one integer i_j; $1 \leqslant i_j \leqslant m$ such that $e_j\bar{R}_{i_j} = 0$. Hence $\bar{R}_{i_j} \subseteq r(e_jR) = T_j$. For each j, i_j is uniquely defined by the above, for otherwise we shall have $T_j = R$. In particular, $k \leqslant m$. Now,

$$\bigcap_{j=1}^{k} \bar{R}_{i_j} \subseteq \bigcap_{j=1}^{k} T_j = 0.$$

Since the intersection of the $\bar{R}_i$ is irredundant, we must have $k = m$. Renumber the T_j, if necessary, to obtain $\bar{R}_j \subseteq T_j$.

We can write $R = f_1R \oplus \cdots \oplus f_kR$, where $f_iR = R_i$ and the f_i are mutually orthogonal central idempotents. We have $T_1 \supseteq R_2 \oplus \cdots \oplus R_k$. Suppose $T_1 \supsetneqq R_2 \oplus \cdots \oplus R_k$. Let $D = T_2 \cap \cdots \cap T_k \neq 0$ (by 7.5). Since $D \not\subseteq T_1$, $f_1D \neq 0$. Similarly, $f_1T_1 \neq 0$. Now f_1D, f_1T_1 are ideals of R_1 and since R_1 is irreducible, $f_1D \cap f_1T_1 \neq 0$. But

$$f_1D \cap f_1T_1 \subseteq T_1 \cap T_2 \cap \cdots \cap T_k = 0,$$

a contradiction. Hence $T_1 = R_2 \oplus \cdots \oplus R_k = R_1'$. Similarly, for other T_i.
(2) By (1) we have $T_i + T_j = R$ for any $i \neq j$. Hence for any r such that $1 \leqslant r \leqslant n$, $r(e_rR) \supseteq T_i$ for exactly one i; $1 \leqslant i \leqslant k$. Relabel the e_i as e_{ij} so that $r(e_{ij}R) \supseteq T_i$ for each j and distinct e's have distinct labels. Set $A_i = \sum_j e_{ij}R$. Then $r(A_i) \supseteq T_i$ by construction. Hence $A_i \subseteq l(T_i) = l(R_i') = R_i$. Since $R = \sum_i A_i = \sum_i R_i$, it follows that $A_i = R_i$ for all i.

The next theorem provides us with a natural class of rings satisfying condition (T).

THEOREM 7.7. *Let R be a hereditary ring which is an order in a QF 2 ring. Then R satisfies condition (T).*

Proof. Let Q be the quotient ring of R. Let U_1 be a uniform right ideal which is a direct summand of Q. Then $U_1 = r_Q(c^{-1}a)$ for some $a, c \in R$; c regular. Hence $U_1 \cap R = r_R(a)$. Now since R is hereditary, the exact sequence $0 \to r(a) \to R \to aR \to 0$ splits. Hence $r(a) = U_1 \cap R$ is a direct summand of R. Let $R = (U_1 \cap R) \oplus A$, where A is a right ideal of R. Then we have $Q = U_1 \oplus AQ$. Let U_2 be a uniform right ideal of Q, which is a direct summand of AQ. Then $U_2 \cap R \subseteq AQ \cap R = A$. As above, we can show that $U_2 \cap R$ is a direct summand of A. Proceeding this way, we obtain an integer n such that $R = (U_1 \cap R) \oplus \cdots \oplus (U_n \cap R)$. Now by 3.6 each $U_i \cap R$ is a uniform right ideal of R. Similarly, we can show that R is also expressible as a direct sum of uniform left ideals.

ACKNOWLEDGMENT

I wish to thank my supervisor Prof. A. W. Goldie for his constant advice and encouragement. I am also indebted to Dr. M. Slater for several suggestions.

REFERENCES

1. R. R. COLBY AND E. A. RUTTER, JR., The structure of certain artinian rings with zero singular ideal, *J. Algebra* 8 (1968), 156–164.
2. C. W. CURTIS AND I. REINER, "Representation Theory of Finite Groups and Associative Algebras," Interscience, New York, 1962.

3. M. DJABALI, Anneau de fractions d'un J-anneau, *Can. J. Math.* **20** (1968), 182–202.
4. M. DWAN, Thesis, University of Leeds, Leeds, England, 1967.
5. A. W. GOLDIE, "Rings with Maximum Condition," Multigraphed Notes, Yale University, Yale, 1961.
6. A. W. GOLDIE, Lectures Given at the Seminar of the Canadian Mathematical Congress, York University, Toronto, 1967.
7. A. W. GOLDIE, Torsion-free modules and rings, *J. Algebra* **1** (1964), 268–287.
8. H. KUPISCH, Beitrage zur Theorie Nichthalbeinfacher Ringe mit Minimalbedingung, *J. Reine Angew. Math.* **201** (1959), 100–112.